

# NEW DISCRETIZATION SCHEMES FOR TIME-HARMONIC MAXWELL EQUATIONS BY WEAK GALERKIN FINITE ELEMENT METHODS

CHUNMEI WANG\*

**Abstract.** This paper introduces new discretization schemes for time-harmonic Maxwell equations in a connected domain by using the weak Galerkin (WG) finite element method. The corresponding WG algorithms are analyzed for their stability and convergence. Error estimates of optimal order in various discrete Sobolev norms are established for the resulting finite element approximations.

**Key words.** weak Galerkin, finite element methods, time-harmonic, Maxwell equations, weak divergence, weak curl, connected domains, polygonal/polyhedral meshes.

**AMS subject classifications.** Primary 65N30, 65N12, 65N15; Secondary 35Q60, 35B45.

**1. Introduction.** This paper is concerned with new developments of numerical methods for time-harmonic Maxwell equations. The time-harmonic Maxwell equations are coupled magnetic and electric equations given by

$$(1.1) \quad \begin{aligned} \nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t}, & \text{in } \Omega, \\ \nabla \times \mathbf{H} &= \frac{\partial \mathbf{D}}{\partial t} + \mathbf{j}, & \text{in } \Omega, \\ \nabla \cdot \mathbf{D} &= \rho, & \text{in } \Omega, \\ \nabla \cdot \mathbf{B} &= 0, & \text{in } \Omega, \end{aligned}$$

with the constitutive relations:

$$\mathbf{B} = \mu \mathbf{H}, \mathbf{j} = \sigma \mathbf{E} + \mathbf{j}_e, \mathbf{D} = \varepsilon \mathbf{E},$$

where  $\Omega$  is an open bounded and connected domain in  $\mathbb{R}^d (d = 2, 3)$  with a Lipschitz continuous boundary  $\Gamma = \partial\Omega$ . Here,  $\mathbf{E}$  is the electric field intensity,  $\mathbf{B}$  is the magnetic flux density,  $\mathbf{H}$  is the magnetic field intensity,  $\mathbf{D}$  is the electric displacement flux density,  $\mathbf{j}$  is the electric current density,  $\mu = \{\mu_{ij}(\mathbf{x})\}_{d \times d}$  is called permeability,  $\rho$  is the charge density,  $\mathbf{j}_e$  is the external current density,  $\sigma$  is real-valued and is known as the electric conductivity, and  $\varepsilon = \{\varepsilon_{ij}(\mathbf{x})\}_{d \times d}$  is the material parameter, and is called permittivity. Additionally,  $\mu, \varepsilon$  are real-valued, symmetric, uniformly positive definite matrices in the domain  $\Omega$ . We assume that  $\mu, \varepsilon$  and  $\sigma$  are piecewise smooth functions in the domain  $\Omega$ .

For time-harmonic fields, where the time dependence is assumed to be harmonic, i.e.,  $\exp(i\omega t)$ , using the constitutive relations, the maxwell equations (1.1) can be

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rewritten for the Fourier transform of the fields as (see [4] for details)

$$(1.2) \quad \nabla \times \mathbf{E} = -i\omega\mu\mathbf{H}, \quad \text{in } \Omega,$$

$$(1.3) \quad \nabla \times \mathbf{H} = i\omega\varepsilon\mathbf{E} + \sigma\mathbf{E} + \mathbf{j}_e, \quad \text{in } \Omega,$$

$$(1.4) \quad \nabla \cdot (\varepsilon\mathbf{E}) = \rho, \quad \text{in } \Omega,$$

$$(1.5) \quad \nabla \cdot (\mu\mathbf{H}) = 0, \quad \text{in } \Omega,$$

where  $\omega$  is a constant in the domain  $\Omega$ .

In the past several decades, the Maxwell equations have been extensively investigated by many researchers. H(curl) conforming finite element method was first introduced by J. Nédélec [12] and was further developed by P. Monk [9]. Houston, Perugia and Schotzau [6, 7, 8, 13, 14] have developed discontinuous Galerkin (DG) finite element methods for the Maxwell equations. Particularly in [8], a mixed DG formulation for the Maxwell equations was introduced and analyzed. Recently, a weakly over-penalized symmetric interior penalty method [2] has been introduced and analyzed by S. Brenner, F. Li and L. Sung. There are also many other numerical methods developed to discretize the Maxwell equations.

Recently, WG method is emerging as an efficient finite element technique for partial differential equations. The WG finite element method was first introduced in [19, 21] for second order elliptic equations and the idea was subsequently further developed for several other model PDEs [10, 15, 16, 17, 18, 20]. The key idea of WG method is to use weak functions and their corresponding discrete weak derivatives in existing variational forms. WG method is highly flexible and robust by allowing the use of discontinuous piecewise polynomials and finite element partitions with arbitrary shape of polygons/polyhedra, and the method is parameter free and absolutely stable. WG finite element method has been applied to time-harmonic Maxwell equations in [11], yielding a numerical method that has optimal order of convergence in certain discrete norms.

The goal of this paper is to present a new WG finite element method for the time-harmonic Maxwell equations (1.2)-(1.5) in a connected domain with heterogeneous media, which covers more cases compared with the model problem considered in [11]. In particular, we formulate the time-harmonic Maxwell equations (1.2)-(1.5) into two variational problems with complex coefficients; see (2.3) and (2.4) for details. Each of the variational problems is then discretized by using the weak Galerkin finite element method. The main difficulty in the design of numerical methods for (2.3) and (2.4) lies in the fact that the terms  $\nabla \cdot (\varepsilon\mathbf{E})$  and  $\nabla \cdot (\mu\mathbf{H})$  require the continuity of  $\varepsilon\mathbf{E}$  and  $\mu\mathbf{H}$  in the normal direction of all interior interfaces, respectively. Consequently, the usual  $H(\text{div})$  or  $H(\text{curl})$  conforming elements are not applicable in this practice. This paper shows that the weak Galerkin finite element method offers an ideal solution, as the continuity can be relaxed by a weak continuity implemented through a carefully chosen stabilizer.

The paper is organized as follows. In Section 2, we shall derive two variational problems: one for the electric field intensity and the other for the magnetic field intensity. These variational problems form the basis of the weak Galerkin finite element methods of this paper. In Section 3, we shall briefly review the discrete weak divergence and the discrete weak curl operators which are necessary in weak Galerkin. In Section 4, we describe how the weak Galerkin finite element algorithms are formulated. Section 5 is devoted to a verification of some stability conditions for the

resulting WG algorithms. In particular, it is shown in this section that the WG algorithms have one and only one solution. In Section 6, we derive some error equations for our WG algorithms. Finally in Section 7, we establish some optimal order error estimates for the WG finite element approximations.

Throughout the paper, we will follow the usual notations for Sobolev spaces and norms [5]. For any open bounded domain  $D \subset \mathbb{R}^d (d = 2, 3)$  with Lipschitz continuous boundary, we use  $\|\cdot\|_{s,D}$  and  $|\cdot|_{s,D}$  to denote the norm and seminorm in the Sobolev space  $H^s(D)$  for any  $s \geq 0$ , respectively. The inner product in  $H^s(D)$  is denoted by  $(\cdot, \cdot)_{s,D}$ . The space  $H^0(D)$  coincides with  $L^2(D)$ , for which the norm and the inner product are denoted by  $\|\cdot\|_D$  and  $(\cdot, \cdot)_D$ , respectively.

We introduce the following Sobolev space

$$H(\operatorname{div}_\varepsilon; D) = \{\mathbf{v} \in [L^2(D)]^d : \nabla \cdot (\varepsilon \mathbf{v}) \in L^2(D)\},$$

with norm given by

$$\|\mathbf{v}\|_{H(\operatorname{div}_\varepsilon; D)} = (\|\mathbf{v}\|_D^2 + \|\nabla \cdot (\varepsilon \mathbf{v})\|_D^2)^{\frac{1}{2}},$$

where  $\nabla \cdot (\varepsilon \mathbf{v})$  is the divergence of  $\varepsilon \mathbf{v}$ . Any  $\mathbf{v} \in H(\operatorname{div}_\varepsilon; D)$  can be assigned a trace for the normal component of  $\varepsilon \mathbf{v}$  on the boundary. Denote the subspace of  $H(\operatorname{div}_\varepsilon; D)$  with vanishing trace in the normal component by

$$H_0(\operatorname{div}_\varepsilon; D) = \{\mathbf{v} \in H(\operatorname{div}_\varepsilon; D) : (\varepsilon \mathbf{v}) \cdot \mathbf{n}|_{\partial D} = 0\}.$$

When  $\varepsilon = I$  is the identity matrix, the spaces  $H(\operatorname{div}_\varepsilon; D)$  and  $H_0(\operatorname{div}_\varepsilon; D)$  are denoted as  $H(\operatorname{div}; D)$  and  $H_0(\operatorname{div}; D)$ , respectively.

We also use the following Sobolev space

$$H(\operatorname{curl}; D) = \{\mathbf{v} : \mathbf{v} \in [L^2(D)]^d, \nabla \times \mathbf{v} \in [L^2(D)]^d\}$$

with norm given by

$$\|\mathbf{v}\|_{H(\operatorname{curl}; D)} = (\|\mathbf{v}\|_D^2 + \|\nabla \times \mathbf{v}\|_D^2)^{\frac{1}{2}},$$

where  $\nabla \times \mathbf{v}$  is the curl of  $\mathbf{v}$ . Any  $\mathbf{v} \in H(\operatorname{curl}; D)$  can be assigned a trace for its tangential component on the boundary. Denote the subspace of  $H(\operatorname{curl}; D)$  with vanishing trace in the tangential component by

$$H_0(\operatorname{curl}; D) = \{\mathbf{v} \in H(\operatorname{curl}; D) : \mathbf{v} \times \mathbf{n}|_{\partial D} = 0\}.$$

When  $D = \Omega$ , we shall drop the subscript  $D$  in the norm and inner product notation. For convenience, throughout the paper, we use “ $\lesssim$ ” to denote “less than or equal to up to a general constant independent of the mesh size or functions appearing in the inequality”.

**2. Variational Formulations.** The goal of this section is to derive two different variational formulations for the time-harmonic Maxwell model problem (1.2)-(1.5).

**2.1. Variational Formulation I.** For the electric field intensity  $\mathbf{E}$ , we first apply the differential operator  $\nabla \times \mu^{-1}$  to (1.2), and then use the equation (1.3) to obtain

$$(2.1) \quad \nabla \times (\mu^{-1} \nabla \times \mathbf{E}) = (\omega^2 \varepsilon - i\omega \sigma) \mathbf{E} - i\omega \mathbf{j}_e, \quad \text{in } \Omega.$$

A typical boundary condition for the electric field intensity  $\mathbf{E}$  is given by

$$(2.2) \quad \mathbf{E} \times \mathbf{n} = 0, \quad \text{on } \Gamma,$$

where  $\mathbf{n}$  is the unit outward normal direction to  $\Gamma$ .

Therefore, a variational formulation for the electric field intensity  $\mathbf{E}$  seeks  $\mathbf{E} \in H_0(\text{curl}; \Omega) \cap H(\text{div}_\varepsilon; \Omega)$  and  $p \in L^2(\Omega)$  such that

$$(2.3) \quad \begin{aligned} (\mu^{-1} \nabla \times \mathbf{E}, \nabla \times \mathbf{v}) + ((i\omega \sigma - \omega^2 \varepsilon) \mathbf{E}, \mathbf{v}) - (\nabla \cdot (\varepsilon \mathbf{v}), p) &= -(i\omega \mathbf{j}_e, \mathbf{v}), \\ (\nabla \cdot (\varepsilon \mathbf{E}), q) &= (\rho, q), \end{aligned}$$

for all  $\mathbf{v} \in H_0(\text{curl}, \Omega) \cap H(\text{div}_\varepsilon, \Omega)$  and  $q \in L^2(\Omega)$ .

**2.2. Variational Formulation II.** For the magnetic field intensity  $\mathbf{H}$ , we apply  $\nabla \times (i\omega \varepsilon + \sigma)^{-1}$  to the equation (1.3), and then use the equation (1.2) to obtain

$$\nabla \times ((i\omega \varepsilon + \sigma)^{-1} \nabla \times \mathbf{H}) = -i\omega \mu \mathbf{H} + \nabla \times (i\omega \varepsilon + \sigma)^{-1} \mathbf{j}_e, \quad \text{in } \Omega.$$

The boundary conditions are

$$\begin{aligned} ((i\omega \varepsilon + \sigma)^{-1} \nabla \times \mathbf{H}) \times \mathbf{n} &= 0, \quad \text{on } \Gamma, \\ \mu \mathbf{H} \cdot \mathbf{n} &= 0, \quad \text{on } \Gamma. \end{aligned}$$

Note that  $\mathbf{j}_e$  as a volume current has no contribution on the boundary  $\Gamma$ .

A variational formulation for the magnetic field intensity  $\mathbf{H}$  seeks  $\mathbf{H} \in H(\text{curl}; \Omega) \cap H_0(\text{div}_\mu; \Omega)$  and  $p \in L_0^2(\Omega)$  such that

$$(2.4) \quad \begin{aligned} ((i\omega \varepsilon + \sigma)^{-1} \nabla \times \mathbf{H}, \nabla \times \mathbf{v}) + (i\omega \mu \mathbf{H}, \mathbf{v}) - (\nabla \cdot (\mu \mathbf{v}), p) &= (\nabla \times (i\omega \varepsilon + \sigma)^{-1} \mathbf{j}_e, \mathbf{v}), \\ (\nabla \cdot (\mu \mathbf{H}), q) &= 0, \end{aligned}$$

for all  $\mathbf{v} \in H(\text{curl}; \Omega) \cap H_0(\text{div}_\mu; \Omega)$  and  $q \in L_0^2(\Omega)$ .

For simplicity, throughout the paper, we assume that  $\mu$ ,  $\sigma$  and  $\varepsilon$  are piecewise constants in the domain  $\Omega$  with respect to the finite element partitions to be specified in forthcoming sections. The results can be extended to piecewise smooth coefficients without any technical difficulties.

**3. Weak Differential Operators.** The variational formulations (2.3) and (2.4) are based on two differential operators: divergence and curl. In this section, we will introduce weak divergence operator for vector-valued functions of the form  $\varepsilon \mathbf{v}$  and then review the definition for the weak curl operator. More details can be found in [15].

Let  $K \subset \Omega$  be any open bounded domain with boundary  $\partial K$ . Denote by  $\mathbf{n}$  the unit outward normal direction on  $\partial K$ . The space of weak vector-valued functions in  $K$  is defined as follows

$$V(K) = \{\mathbf{v} = \{\mathbf{v}_0, \mathbf{v}_b\} : \mathbf{v}_0 \in [L^2(K)]^d, \mathbf{v}_b \in [L^2(\partial K)]^d\},$$

where  $\mathbf{v}_0$  represents the value of  $\mathbf{v}$  in the interior of  $K$ , and  $\mathbf{v}_b$  the information of  $\mathbf{v}$  on the boundary  $\partial K$ . There are two piece of information of  $\mathbf{v}$  on  $\partial K$  which are needed in the variational formulations (2.3) and (2.4): one of them is the tangential component  $\mathbf{n} \times (\mathbf{v} \times \mathbf{n})$  and the other one is the normal component of  $\varepsilon \mathbf{v}$  on  $\partial K$  given by  $(\varepsilon \mathbf{v} \cdot \mathbf{n})\mathbf{n}$ . Intuitively, the vector  $\mathbf{v}_b$  is used to represent both of them as follows

$$(3.1) \quad \mathbf{v}_b = (\varepsilon \mathbf{v} \cdot \mathbf{n})\mathbf{n} + \mathbf{n} \times (\mathbf{v} \times \mathbf{n}).$$

We emphasize that the right-hand side of (3.1) is not meant to be a decomposition of the trace of  $\mathbf{v}$  on  $\partial K$ .

**3.1. Weak divergence and discrete weak divergence [15, 20].** For any  $\mathbf{v} \in V(K)$ , the weak divergence of  $\varepsilon \mathbf{v}$ , denoted by  $\nabla_{w,K} \cdot (\varepsilon \mathbf{v})$ , is defined as a bounded linear functional on the Sobolev space  $H^1(K)$  satisfying

$$\langle \nabla_{w,K} \cdot (\varepsilon \mathbf{v}), \varphi \rangle_K = -(\varepsilon \mathbf{v}_0, \nabla \varphi)_K + \langle \mathbf{v}_b \cdot \mathbf{n}, \varphi \rangle_{\partial K}, \quad \forall \varphi \in H^1(K).$$

Here the left-hand side stands for the action of the linear functional on  $\varphi \in H^1(K)$ , and  $\langle \cdot, \cdot \rangle_{\partial K}$  is the inner product in  $L^2(\partial K)$ . The discrete weak divergence of  $\varepsilon \mathbf{v}$ , denoted by  $\nabla_{w,r,K} \cdot (\varepsilon \mathbf{v})$ , is defined as the unique polynomial in  $P_r(K)$ ,  $r \geq 0$ , satisfying

$$(3.2) \quad (\nabla_{w,r,K} \cdot (\varepsilon \mathbf{v}), \varphi)_K = -(\varepsilon \mathbf{v}_0, \nabla \varphi)_K + \langle \mathbf{v}_b \cdot \mathbf{n}, \varphi \rangle_{\partial K}, \quad \forall \varphi \in P_r(K),$$

where  $P_r(K)$  is the set of all polynomials on  $K$  with degree  $r$  or less.

Assume that  $\mathbf{v}_0$  is sufficiently smooth such that  $\nabla \cdot (\varepsilon \mathbf{v}_0) \in L^2(K)$ . By applying the integration by parts to the first term on the right-hand side of (3.2), we have

$$(3.3) \quad (\nabla_{w,r,K} \cdot (\varepsilon \mathbf{v}), \varphi)_K = (\nabla \cdot (\varepsilon \mathbf{v}_0), \varphi)_K + \langle (\mathbf{v}_b - \varepsilon \mathbf{v}_0) \cdot \mathbf{n}, \varphi \rangle_{\partial K},$$

for any  $\varphi \in P_r(K)$ .

**3.2. Weak curl and discrete weak curl [11, 15].** The weak curl of  $\mathbf{v} \in V(K)$ , denoted by  $\nabla_{w,K} \times \mathbf{v}$ , is defined as a bounded linear functional on the Sobolev space  $[H^1(K)]^d$  satisfying

$$\langle \nabla_{w,K} \times \mathbf{v}, \varphi \rangle_K = (\mathbf{v}_0, \nabla \times \varphi)_K - \langle \mathbf{v}_b \times \mathbf{n}, \varphi \rangle_{\partial K}, \quad \forall \varphi \in [H^1(K)]^d.$$

The discrete weak curl of  $\mathbf{v} \in V(K)$ , denoted by  $\nabla_{w,r,K} \times \mathbf{v}$ , is defined as the unique polynomial-valued vector in  $[P_r(K)]^d$ , such that

$$(3.4) \quad (\nabla_{w,r,K} \times \mathbf{v}, \varphi)_K = (\mathbf{v}_0, \nabla \times \varphi)_K - \langle \mathbf{v}_b \times \mathbf{n}, \varphi \rangle_{\partial K}, \quad \forall \varphi \in [P_r(K)]^d.$$

For sufficiently smooth  $\mathbf{v}_0$  with  $\nabla \times \mathbf{v}_0 \in [L^2(K)]^d$ , by applying the integration by parts to the first term on the right-hand side of (3.4), we obtain

$$(3.5) \quad (\nabla_{w,r,K} \times \mathbf{v}, \varphi)_K = (\nabla \times \mathbf{v}_0, \varphi)_K - \langle (\mathbf{v}_b - \mathbf{v}_0) \times \mathbf{n}, \varphi \rangle_{\partial K},$$

for any  $\varphi \in [P_r(K)]^d$ .

**REMARK 3.1.** All the definitions and formulations with respect to the coefficient  $\varepsilon$  of this section can be generalized to the coefficient  $\mu$ . This is particularly useful in the study of the equation for the magnetic field intensity function.

**4. Numerical Algorithms by Weak Galerkin.** Let  $\mathcal{T}_h$  be a finite element partition of the domain  $\Omega \subset \mathbb{R}^d (d = 2, 3)$  with mesh size  $h$ . Assume that  $\mathcal{T}_h$  consists of polygons/polyhedra of arbitrary shape and is shape regular as defined in [19]. Denote by  $\mathcal{E}_h$  the set of all edges/faces in  $\mathcal{T}_h$  and  $\mathcal{E}_h^0 = \mathcal{E}_h \setminus \partial\Omega$  the set of all interior edges/faces in  $\mathcal{T}_h$ . For each interior edge/face  $e \in \mathcal{E}_h^0$ , we assign a prescribed normal direction  $\mathbf{n}_e$  to  $e$ . Denote by  $\mathbf{n}$  the unit outward normal direction to the boundary  $\Gamma$ . Denote the jump of  $q$  on the edge/face  $e \in \mathcal{E}_h$  by

$$(4.1) \quad \llbracket q \rrbracket = \begin{cases} q|_{\partial T_1} - q|_{\partial T_2}, & e \in \mathcal{E}_h^0, \\ q, & e \subset \partial\Omega, \end{cases}$$

where  $q|_{\partial T_i}$  denotes the value of  $q$  on an edge/face  $e$  as seen from the element  $T_i$ ,  $i = 1, 2$ . Here  $T_1$  and  $T_2$  are the two elements that share  $e$  as a common edge/face. The order of  $T_1$  and  $T_2$  is non-essential in (4.1) as long as the difference is taken in a consistent way in all the formulas. If  $e \subset \Gamma$  is a boundary edge, then  $\llbracket q \rrbracket = q|_e$  is defined as its trace on  $e$ .

Let  $k \geq 1$  be a given integer. For each element  $T \in \mathcal{T}_h$ , we define the local weak finite element space by

$$\mathbf{V}(k, T) = \{\mathbf{v} = \{\mathbf{v}_0, \mathbf{v}_b\} : \mathbf{v}_0 \in [P_k(T)]^d, \mathbf{v}_b \in [P_k(e)]^d, e \in (\partial T \cap \mathcal{E}_h)\}.$$

By patching the local weak finite element space  $\mathbf{V}(k, T)$  together over all the elements  $T \in \mathcal{T}_h$  through a common value  $\mathbf{v}_b$  on the interior interface  $\mathcal{E}_h^0$ , we obtain a global weak finite element space:

$$(4.2) \quad \mathbf{V}_h = \{\mathbf{v} = \{\mathbf{v}_0, \mathbf{v}_b\} : \mathbf{v}|_T \in \mathbf{V}(k, T), T \in \mathcal{T}_h\}.$$

Introduce two subspaces of  $\mathbf{V}_h$  as follows:

$$\mathbf{V}_{h,0}^1 = \{\mathbf{v} = \{\mathbf{v}_0, \mathbf{v}_b\} \in \mathbf{V}_h : \mathbf{v}_b \times \mathbf{n} = 0 \text{ on } \Gamma\},$$

$$\mathbf{V}_{h,0}^2 = \{\mathbf{v} = \{\mathbf{v}_0, \mathbf{v}_b\} \in \mathbf{V}_h : \mathbf{v}_b \cdot \mathbf{n} = 0 \text{ on } \Gamma\}.$$

We further define two more finite element spaces

$$W_h^1 = \{q : q \in L^2(\Omega), q|_T \in P_{k-1}(T), T \in \mathcal{T}_h\},$$

$$W_h^2 = \{q : q \in L_0^2(\Omega), q|_T \in P_{k-1}(T), T \in \mathcal{T}_h\}.$$

The discrete weak divergence  $(\nabla_{w,k-1} \cdot)$  and the discrete weak curl  $(\nabla_{w,k-1} \times)$  can be computed by using (3.2) and (3.4) on each element; i.e.,

$$\begin{aligned} (\nabla_{w,k-1} \cdot (\varepsilon \mathbf{v}))|_T &= \nabla_{w,k-1,T} \cdot (\varepsilon \mathbf{v}|_T), & \mathbf{v} \in \mathbf{V}_h, \\ (\nabla_{w,k-1} \cdot (\mu \mathbf{v}))|_T &= \nabla_{w,k-1,T} \cdot (\mu \mathbf{v}|_T), & \mathbf{v} \in \mathbf{V}_h, \\ (\nabla_{w,k-1} \times \mathbf{v})|_T &= \nabla_{w,k-1,T} \times (\mathbf{v}|_T), & \mathbf{v} \in \mathbf{V}_h. \end{aligned}$$

For simplicity of notation and without confusion, we shall drop the subscript  $k-1$  from the notations  $(\nabla_{w,k-1} \cdot)$  and  $(\nabla_{w,k-1} \times)$  from now on.

We introduce the following bilinear forms

$$\begin{aligned}
a_1(\mathbf{v}, \mathbf{w}) &= \sum_{T \in \mathcal{T}_h} (\mu^{-1} \nabla_w \times \mathbf{v}, \nabla_w \times \mathbf{w})_T + ((i\omega\sigma - \omega^2\varepsilon) \mathbf{v}_0, \mathbf{w}_0)_T + s_1(\mathbf{v}, \mathbf{w}), \\
a_2(\mathbf{v}, \mathbf{w}) &= \sum_{T \in \mathcal{T}_h} ((i\omega\varepsilon + \sigma)^{-1} \nabla_w \times \mathbf{v}, \nabla_w \times \mathbf{w})_T + (i\omega\mu \mathbf{v}_0, \mathbf{w}_0)_T + s_2(\mathbf{v}, \mathbf{w}), \\
b_1(\mathbf{v}, q) &= \sum_{T \in \mathcal{T}_h} (\nabla_w \cdot (\varepsilon \mathbf{v}), q)_T, \\
b_2(\mathbf{v}, q) &= \sum_{T \in \mathcal{T}_h} (\nabla_w \cdot (\mu \mathbf{v}), q)_T,
\end{aligned}$$

where

$$\begin{aligned}
s_1(\mathbf{v}, \mathbf{w}) &= \sum_{T \in \mathcal{T}_h} h_T^{-1} \langle (\varepsilon \mathbf{v}_0 - \mathbf{v}_b) \cdot \mathbf{n}, (\varepsilon \mathbf{w}_0 - \mathbf{w}_b) \cdot \mathbf{n} \rangle_{\partial T} + h_T^{-1} \langle (\mathbf{v}_0 - \mathbf{v}_b) \times \mathbf{n}, (\mathbf{w}_0 - \mathbf{w}_b) \times \mathbf{n} \rangle_{\partial T}, \\
s_2(\mathbf{v}, \mathbf{w}) &= \sum_{T \in \mathcal{T}_h} h_T^{-1} \langle (\mu \mathbf{v}_0 - \mathbf{v}_b) \cdot \mathbf{n}, (\mu \mathbf{w}_0 - \mathbf{w}_b) \cdot \mathbf{n} \rangle_{\partial T} + h_T^{-1} \langle (\mathbf{v}_0 - \mathbf{v}_b) \times \mathbf{n}, (\mathbf{w}_0 - \mathbf{w}_b) \times \mathbf{n} \rangle_{\partial T}.
\end{aligned}$$

**WEAK GALERKIN ALGORITHM 1.** *For a numerical approximation of the electric field intensity  $\mathbf{E}$ , one may seek  $\mathbf{E}_h \in \mathbf{V}_{h,0}^1$  and an auxiliary function  $p_h \in W_h^1$ , such that*

$$(4.3) \quad a_1(\mathbf{E}_h, \mathbf{v}) - b_1(\mathbf{v}, p_h) = -(i\omega \mathbf{j}_e, \mathbf{v}_0), \quad \forall \mathbf{v} = \{\mathbf{v}_0, \mathbf{v}_b\} \in \mathbf{V}_{h,0}^1,$$

$$(4.4) \quad b_1(\mathbf{E}_h, w) = (\rho, w), \quad \forall w \in W_h^1.$$

**WEAK GALERKIN ALGORITHM 2.** *For a numerical approximation of the magnetic field intensity  $\mathbf{H}$ , one may seek  $\mathbf{H}_h \in \mathbf{V}_{h,0}^2$  and an auxiliary function  $p_h \in W_h^2$ , such that*

$$(4.5) \quad a_2(\mathbf{H}_h, \mathbf{v}) - b_2(\mathbf{v}, p_h) = (\nabla \times (i\omega\varepsilon + \sigma)^{-1} \mathbf{j}_e, \mathbf{v}_0), \quad \forall \mathbf{v} = \{\mathbf{v}_0, \mathbf{v}_b\} \in \mathbf{V}_{h,0}^2,$$

$$(4.6) \quad b_2(\mathbf{H}_h, w) = 0, \quad \forall w \in W_h^2.$$

**5. Verification of Stability Conditions.** We first introduce two norms: one in the weak finite element space  $\mathbf{V}_{h,0}^1$  and the other in  $\mathbf{V}_{h,0}^2$  as follows:

$$\begin{aligned}
\|\mathbf{v}\|_{\mathbf{V}_{h,0}^1} &= \left( \sum_{T \in \mathcal{T}_h} \|\nabla_w \times \mathbf{v}\|_T^2 + \|\mathbf{v}_0\|_T^2 \right. \\
&\quad \left. + h_T^{-1} \|\varepsilon \mathbf{v}_0 \cdot \mathbf{n} - \mathbf{v}_b \cdot \mathbf{n}\|_{\partial T}^2 + h_T^{-1} \|\mathbf{v}_0 \times \mathbf{n} - \mathbf{v}_b \times \mathbf{n}\|_{\partial T}^2 \right)^{\frac{1}{2}}, \quad \forall \mathbf{v} \in \mathbf{V}_{h,0}^1,
\end{aligned}
\tag{5.1}$$

$$\begin{aligned}
\|\mathbf{v}\|_{\mathbf{V}_{h,0}^2} &= \left( \sum_{T \in \mathcal{T}_h} \|\nabla_w \times \mathbf{v}\|_T^2 + \|\mathbf{v}_0\|_T^2 \right. \\
&\quad \left. + h_T^{-1} \|\mu \mathbf{v}_0 \cdot \mathbf{n} - \mathbf{v}_b \cdot \mathbf{n}\|_{\partial T}^2 + h_T^{-1} \|\mathbf{v}_0 \times \mathbf{n} - \mathbf{v}_b \times \mathbf{n}\|_{\partial T}^2 \right)^{\frac{1}{2}}, \quad \forall \mathbf{v} \in \mathbf{V}_{h,0}^2.
\end{aligned}
\tag{5.2}$$

In the finite element spaces  $W_h^1$  and  $W_h^2$ , we introduce mesh-dependent norms as follows

$$(5.3) \quad \|q\|_{W_h^1} = \left( h^2 \sum_{T \in \mathcal{T}_h} (\varepsilon \nabla q, \nabla q)_T + h \sum_{e \in \mathcal{E}_h} \|\llbracket q \rrbracket\|_e^2 \right)^{\frac{1}{2}}, \quad \forall q \in W_h^1,$$

$$(5.4) \quad \|q\|_{W_h^2} = \left( h^2 \sum_{T \in \mathcal{T}_h} (\mu \nabla q, \nabla q)_T + h \sum_{e \in \mathcal{E}_h^0} \|\llbracket q \rrbracket\|_e^2 \right)^{\frac{1}{2}}, \quad \forall q \in W_h^2.$$

The following two lemmas are concerned with the coercivity of the bilinear forms  $a_1(\cdot, \cdot)$  and  $a_2(\cdot, \cdot)$ . The boundedness of these two bilinear forms is straightforward.

LEMMA 5.1. *There exists a positive constant  $C$  such that for any  $\mathbf{v} \in \mathbf{V}_{h,0}^1$  one has*

$$(5.5) \quad |a_1(\mathbf{v}, \mathbf{v})| \geq C \|\mathbf{v}\|_{\mathbf{V}_{h,0}^1}^2.$$

*Proof.* From the definition of the bilinear form  $a_1(\cdot, \cdot)$  we have

$$a_1(\mathbf{v}, \mathbf{v}) = \sum_{T \in \mathcal{T}_h} (\mu^{-1} \nabla_w \times \mathbf{v}, \nabla_w \times \mathbf{v})_T + ((i\omega\sigma - \omega^2\varepsilon)\mathbf{v}_0, \mathbf{v}_0)_T + s_1(\mathbf{v}, \mathbf{v}).$$

Since imaginary part of  $a_1(\mathbf{v}, \mathbf{v})$  is given by  $(\omega\sigma\mathbf{v}_0, \mathbf{v}_0)$ , then we have

$$(5.6) \quad \omega\sigma_0 \|\mathbf{v}_0\|^2 \leq (\omega\sigma\mathbf{v}_0, \mathbf{v}_0) \leq |a_1(\mathbf{v}, \mathbf{v})|,$$

where  $\sigma_0$  is the minimum value of  $\sigma$  over  $\Omega$ . The real part of  $a_1(\mathbf{v}, \mathbf{v})$  is given by

$$\operatorname{Re}(a_1(\mathbf{v}, \mathbf{v})) = \sum_{T \in \mathcal{T}_h} (\mu^{-1} \nabla_w \times \mathbf{v}, \nabla_w \times \mathbf{v})_T - (\omega^2\varepsilon\mathbf{v}_0, \mathbf{v}_0)_T + s_1(\mathbf{v}, \mathbf{v}).$$

Thus,

$$(5.7) \quad \left| \sum_{T \in \mathcal{T}_h} (\mu^{-1} \nabla_w \times \mathbf{v}, \nabla_w \times \mathbf{v})_T - (\omega^2\varepsilon\mathbf{v}_0, \mathbf{v}_0)_T + s_1(\mathbf{v}, \mathbf{v}) \right| \leq |a_1(\mathbf{v}, \mathbf{v})|.$$

Combining (5.6) with (5.7) gives rise to the coercivity estimate (5.5). This completes the proof of the lemma.  $\square$

LEMMA 5.2. *There exists a positive constant  $C$  such that for any  $\mathbf{v} \in \mathbf{V}_{h,0}^2$  one has*

$$(5.8) \quad |a_2(\mathbf{v}, \mathbf{v})| \geq C \|\mathbf{v}\|_{\mathbf{V}_{h,0}^2}^2.$$

*Proof.* From the definition of the bilinear form  $a_2(\cdot, \cdot)$  we have

$$a_2(\mathbf{v}, \mathbf{v}) = \sum_{T \in \mathcal{T}_h} ((i\omega\varepsilon + \sigma)^{-1} \nabla_w \times \mathbf{v}, \nabla_w \times \mathbf{v})_T + (i\omega\mu\mathbf{v}_0, \mathbf{v}_0)_T + s_2(\mathbf{v}, \mathbf{v})$$



The real part of  $a_2(\mathbf{v}, \mathbf{v})$  is given by

$$\operatorname{Re}(a_2(\mathbf{v}, \mathbf{v})) = \sum_{T \in \mathcal{T}_h} (\sigma(\sigma^2 + \omega^2 \varepsilon^2)^{-1} \nabla_w \times \mathbf{v}, \nabla_w \times \mathbf{v})_T + s_2(\mathbf{v}, \mathbf{v}) \geq 0.$$

Hence, we have

$$(5.9) \quad \sum_{T \in \mathcal{T}_h} (\sigma(\sigma^2 + \omega^2 \varepsilon^2)^{-1} \nabla_w \times \mathbf{v}, \nabla_w \times \mathbf{v})_T + s_2(\mathbf{v}, \mathbf{v}) \leq |a_2(\mathbf{v}, \mathbf{v})|.$$

The imaginary part of  $a_2(\mathbf{v}, \mathbf{v})$  is given by

$$\operatorname{Im}(a_2(\mathbf{v}, \mathbf{v})) = \sum_{T \in \mathcal{T}_h} (\omega \mu \mathbf{v}_0, \mathbf{v}_0)_T - (\omega \varepsilon (\sigma^2 + \omega^2 \varepsilon^2)^{-1} \nabla_w \times \mathbf{v}, \nabla_w \times \mathbf{v})_T,$$

which leads to

$$(5.10) \quad \left| \sum_{T \in \mathcal{T}_h} (\omega \mu \mathbf{v}_0, \mathbf{v}_0)_T - (\omega \varepsilon (\sigma^2 + \omega^2 \varepsilon^2)^{-1} \nabla_w \times \mathbf{v}, \nabla_w \times \mathbf{v})_T \right| \leq |a_2(\mathbf{v}, \mathbf{v})|.$$

Combining (5.9) with (5.10) gives rise to the coercivity estimate (5.8). This completes the proof of the lemma.  $\square$

Next, we establish an *inf-sup* condition for the bilinear form  $b_1(\cdot, \cdot)$  used in the WG algorithm 1. To this end, for any  $q \in W_h^1$ , set  $\mathbf{v}_q = \{-h^2 \nabla q; h \mathbf{v}_{q,b}\} \in \mathbf{V}_{h,0}^1$ , where

$$(5.11) \quad \mathbf{v}_{q,b} = \begin{cases} \llbracket q \rrbracket \mathbf{n}_e, & \text{on } e \in \mathcal{E}_h^0, \\ q \mathbf{n}, & \text{on } e \in \mathcal{E}_h \cap \Gamma. \end{cases}$$

Now for any  $\mathbf{v} = \{\mathbf{v}_0; \mathbf{v}_b\} \in \mathbf{V}_{h,0}^1$ , from the definition (3.2) of weak divergence, we have

$$(5.12) \quad \begin{aligned} b_1(\mathbf{v}, q) &= \sum_{T \in \mathcal{T}_h} (\nabla_w \cdot (\varepsilon \mathbf{v}), q)_T \\ &= \sum_{T \in \mathcal{T}_h} -(\varepsilon \mathbf{v}_0, \nabla q)_T + \langle \mathbf{v}_b \cdot \mathbf{n}, q \rangle_{\partial T} \\ &= - \sum_{T \in \mathcal{T}_h} (\varepsilon \mathbf{v}_0, \nabla q)_T + \sum_{e \in \mathcal{E}_h} \langle \mathbf{v}_b \cdot \mathbf{n}_e, \llbracket q \rrbracket \rangle_e. \end{aligned}$$

LEMMA 5.3. (inf-sup condition for WG algorithm 1) *For any  $q \in W_h^1$ , there exists a finite element function  $\mathbf{v}_q \in \mathbf{V}_{h,0}^1$  such that*

$$(5.13) \quad b_1(\mathbf{v}_q, q) = h^2 \sum_{T \in \mathcal{T}_h} (\varepsilon \nabla q, \nabla q)_T + h \sum_{e \in \mathcal{E}_h} \|\llbracket q \rrbracket\|_e^2,$$

$$(5.14) \quad \|\mathbf{v}_q\|_{\mathbf{V}_{h,0}^1} \lesssim \|q\|_{W_h^1}.$$

*Proof.* For any  $q \in W_h^1$ , we define  $\mathbf{v}_{q,b}$  by (5.11) and set  $\mathbf{v}_q = \{-h^2 \nabla q; h \mathbf{v}_{q,b}\} \in \mathbf{V}_{h,0}^1$ . By letting  $\mathbf{v} = \mathbf{v}_q$  in (5.12) we obtain

$$b_1(\mathbf{v}_q, q) = h^2 \sum_{T \in \mathcal{T}_h} (\varepsilon \nabla q, \nabla q)_T + h \sum_{e \in \mathcal{E}_h} \|\llbracket q \rrbracket\|_e^2,$$

which verifies the identity (5.13).

To derive (5.14), we consider the following decomposition

$$\mathbf{v}_q = \mathbf{v}_q^{(1)} + \mathbf{v}_q^{(2)},$$

where  $\mathbf{v}_q^{(1)} = -\{h^2 \nabla q; 0\}$  and  $\mathbf{v}_q^{(2)} = \{0; h \mathbf{v}_{q,b}\}$ . It suffices to establish (5.14) for  $\mathbf{v}_q^{(1)}$  and  $\mathbf{v}_q^{(2)}$  respectively. Using (5.1), we have

$$(5.15) \quad \begin{aligned} \|\mathbf{v}_q^{(1)}\|_{\mathbf{V}_{h,0}^1}^2 &= \sum_{T \in \mathcal{T}_h} \|\nabla_w \times \mathbf{v}_q^{(1)}\|_T^2 + \|h^2 \nabla q\|_T^2 \\ &\quad + h_T^{-1} \|h^2 \varepsilon \nabla q \cdot \mathbf{n}\|_{\partial T}^2 + h_T^{-1} \|h^2 \nabla q \times \mathbf{n}\|_{\partial T}^2. \end{aligned}$$

It follows from (3.4) of the discrete weak curl that

$$(\nabla_w \times \mathbf{v}_q^{(1)}, \varphi)_T = -h^2 (\nabla q, \nabla \times \varphi)_T, \quad \forall \varphi \in [P_{k-1}(T)]^d.$$

Using the inverse inequality (7.2) we obtain

$$\|\nabla_w \times \mathbf{v}_q^{(1)}\|_T \lesssim h \|\nabla q\|_T.$$

Substituting the above inequality into (5.15) and then using the trace inequality (7.3) gives rise to

$$\|\mathbf{v}_q^{(1)}\|_{\mathbf{V}_{h,0}^1}^2 \lesssim h^2 \|\nabla q\|_T^2,$$

which verifies the estimate (5.14) for  $\mathbf{v}_q^{(1)}$ .

For  $\mathbf{v}_q^{(2)}$ , we again use (5.1) to obtain

$$(5.16) \quad \|\mathbf{v}_q^{(2)}\|_{\mathbf{V}_{h,0}^1}^2 = \sum_{T \in \mathcal{T}_h} \|\nabla_w \times \mathbf{v}_q^{(2)}\|_T^2 + h_T^{-1} \|h \mathbf{v}_{q,b} \cdot \mathbf{n}\|_{\partial T}^2 + h_T^{-1} \|h \mathbf{v}_{q,b} \times \mathbf{n}\|_{\partial T}^2.$$

Since  $\mathbf{v}_{q,b}$  is parallel to  $\mathbf{n}$ , then  $\mathbf{v}_{q,b} \times \mathbf{n} = 0$  on  $\partial T$ . In addition, (3.4) of the discrete weak curl implies  $\nabla_w \times \mathbf{v}_q^{(2)} = 0$ , since

$$(\nabla_w \times \mathbf{v}_q^{(2)}, \varphi)_T = (0, \nabla \times \varphi)_T - h \langle \mathbf{v}_{q,b} \times \mathbf{n}, \varphi \rangle_{\partial T} = 0, \quad \forall \varphi \in [P_{k-1}(T)]^d.$$

Thus, it follows from (5.16) and (5.11) that

$$\|\mathbf{v}_q^{(2)}\|_{\mathbf{V}_{h,0}^1}^2 \lesssim h \sum_{e \in \mathcal{E}_h} \|\llbracket q \rrbracket\|_e^2,$$

which verifies the estimate (5.14) for  $\mathbf{v}_q^{(2)}$ . This completes the proof of the lemma.  $\square$

For the bilinear form  $b_2(\cdot, \cdot)$ , we may follow the same spirit of Lemma 5.3 to derive an *inf-sup* condition. For completeness, we present all the necessary details as follows. For any  $q \in W_h^2$ , define a finite element function  $\mathbf{v}_q = \{-h^2 \nabla q; h \mathbf{v}_{q,b}\} \in \mathbf{V}_{h,0}^2$ , where

$$(5.17) \quad \mathbf{v}_{q,b} = \begin{cases} \llbracket q \rrbracket \mathbf{n}_e, & \text{on } e \in \mathcal{E}_h^0, \\ 0, & \text{on } e \in \mathcal{E}_h \cap \Gamma. \end{cases}$$

Note that for any  $\mathbf{v} = \{\mathbf{v}_0; \mathbf{v}_b\} \in \mathbf{V}_{h,0}^2$ , from (3.2) of weak divergence, we have

$$\begin{aligned}
b_2(\mathbf{v}, q) &= \sum_{T \in \mathcal{T}_h} (\nabla_w \cdot (\mu \mathbf{v}), q)_T \\
(5.18) \quad &= \sum_{T \in \mathcal{T}_h} -(\mu \mathbf{v}_0, \nabla q)_T + \langle \mathbf{v}_b \cdot \mathbf{n}, q \rangle_{\partial T} \\
&= - \sum_{T \in \mathcal{T}_h} (\mu \mathbf{v}_0, \nabla q)_T + \sum_{e \in \mathcal{E}_h^0} \langle \mathbf{v}_b \cdot \mathbf{n}_e, \llbracket q \rrbracket \rangle_e.
\end{aligned}$$

LEMMA 5.4. (inf-sup condition for WG algorithm 2) *For any  $q \in W_h^2$ , there exists a finite element function  $\mathbf{v}_q \in \mathbf{V}_{h,0}^2$  such that*

$$(5.19) \quad b_2(\mathbf{v}_q, q) = h^2 \sum_{T \in \mathcal{T}_h} (\mu \nabla q, \nabla q)_T + h \sum_{e \in \mathcal{E}_h^0} \|\llbracket q \rrbracket\|_e^2,$$

$$(5.20) \quad \|\mathbf{v}_q\|_{\mathbf{V}_{h,0}^2} \lesssim \|q\|_{W_h^2}.$$

*Proof.* For any  $q \in W_h^2$ , we define  $\mathbf{v}_{q,b}$  by (5.17) and set  $\mathbf{v}_q = \{-h^2 \nabla q; h \mathbf{v}_{q,b}\}$ . It is easy to see that  $\mathbf{v}_q \in \mathbf{V}_{h,0}^2$ . By letting  $\mathbf{v} = \mathbf{v}_q$  in (5.18) we arrive at

$$b_2(\mathbf{v}_q, q) = h^2 \sum_{T \in \mathcal{T}_h} (\mu \nabla q, \nabla q)_T + h \sum_{e \in \mathcal{E}_h^0} \|\llbracket q \rrbracket\|_e^2,$$

which verifies the identity (5.19).

To derive (5.20), we consider the following decomposition

$$\mathbf{v}_q = \mathbf{v}_q^{(1)} + \mathbf{v}_q^{(2)},$$

where  $\mathbf{v}_q^{(1)} = -\{h^2 \nabla q; 0\}$  and  $\mathbf{v}_q^{(2)} = \{0; h \mathbf{v}_{q,b}\}$ . It suffices to establish (5.20) for  $\mathbf{v}_q^{(1)}$  and  $\mathbf{v}_q^{(2)}$  respectively. From (5.2), we have

$$\begin{aligned}
(5.21) \quad \|\mathbf{v}_q^{(1)}\|_{\mathbf{V}_{h,0}^2}^2 &= \sum_{T \in \mathcal{T}_h} \|\nabla_w \times \mathbf{v}_q^{(1)}\|_T^2 + \|h^2 \nabla q\|_T^2 \\
&\quad + h_T^{-1} \|h^2 \mu \nabla q \cdot \mathbf{n}\|_{\partial T}^2 + h_T^{-1} \|h^2 \nabla q \times \mathbf{n}\|_{\partial T}^2.
\end{aligned}$$

The definition (3.4) for the discrete weak curl implies

$$(\nabla_w \times \mathbf{v}_q^{(1)}, \varphi)_T = -h^2 (\nabla q, \nabla \times \varphi)_T, \quad \forall \varphi \in [P_{k-1}(T)]^d.$$

It follows from the inverse inequality (7.2) that

$$\|\nabla_w \times \mathbf{v}_q^{(1)}\|_T \lesssim h \|\nabla q\|_T.$$

Substituting the above into (5.21) and then using the trace inequality (7.3) yields

$$\|\mathbf{v}_q^{(1)}\|_{\mathbf{V}_{h,0}^2}^2 \lesssim h^2 \|\nabla q\|_T^2,$$

which verifies the estimate (5.20) for  $\mathbf{v}_q^{(1)}$ .

For  $\mathbf{v}_q^{(2)}$ , we again use (5.2) to obtain

$$(5.22) \quad \|\mathbf{v}_q^{(2)}\|_{\mathbf{V}_{h,0}^2}^2 = \sum_{T \in \mathcal{T}_h} \|\nabla_w \times \mathbf{v}_q^{(2)}\|_T^2 + h_T^{-1} \|h \mathbf{v}_{q,b} \cdot \mathbf{n}\|_{\partial T}^2 + h_T^{-1} \|h \mathbf{v}_{q,b} \times \mathbf{n}\|_{\partial T}^2.$$

Since  $\mathbf{v}_{q,b}$  is parallel to  $\mathbf{n}$ , then  $\mathbf{v}_{q,b} \times \mathbf{n} = 0$  on  $\partial T$ . In addition, the definition (3.4) for the discrete weak curl implies  $\nabla_w \times \mathbf{v}_q^{(2)} = 0$ , since

$$(\nabla_w \times \mathbf{v}_q^{(2)}, \varphi)_T = (0, \nabla \times \varphi)_T - h \langle \mathbf{v}_{q,b} \times \mathbf{n}, \varphi \rangle_{\partial T} = 0, \quad \forall \varphi \in [P_{k-1}(T)]^d.$$

Thus, it follows from (5.22) and (5.17) that

$$\|\mathbf{v}_q^{(2)}\|_{\mathbf{V}_{h,0}^2}^2 \lesssim h \sum_{e \in \mathcal{E}_h^0} \|\llbracket q \rrbracket_e\|_e^2,$$

which verifies the estimate (5.20) for  $\mathbf{v}_q^{(2)}$ . This completes the proof of the lemma.  $\square$

Using the general result of Babuška [1] and Brezzi [3] we obtain the following result on the solution existence and uniqueness for our WG finite element algorithms.

**THEOREM 5.5.** *The weak Galerkin algorithm 1 or the system of equations (4.3)-(4.4) has a unique solution. The same conclusion can be drawn for the weak Galerkin algorithm 2 or the system of equations (4.5)-(4.6).*

**6. Error Equations.** In this section we shall establish two error equations for the weak Galerkin algorithms 1 and 2. These error equations will be used for deriving error estimates for the resulting numerical schemes.

Let  $Q_0$  be the  $L^2$  projection onto  $[P_k(T)]^d$ ,  $T \in \mathcal{T}_h$ , and  $Q_b$  be the  $L^2$  projection onto  $[P_k(e)]^d$ ,  $e \in \partial T \cap \mathcal{E}_h$ . Denote by  $Q_h$  the  $L^2$  projection onto the weak finite element space  $\mathbf{V}_h$  such that on each element  $T \in \mathcal{T}_h$ ,

$$(6.1) \quad (Q_h \mathbf{u})|_T = \{Q_0 \mathbf{u}, Q_b \mathbf{u}\},$$

where

$$(6.2) \quad Q_b \mathbf{u} = Q_b(\varepsilon \mathbf{u} \cdot \mathbf{n}) \mathbf{n} + Q_b(\mathbf{n} \times (\mathbf{u} \times \mathbf{n})).$$

Observe that  $\mathbf{n} \times (\mathbf{u} \times \mathbf{n}) = \mathbf{u} - (\mathbf{u} \cdot \mathbf{n}) \mathbf{n}$  is the tangential component of the vector  $\mathbf{u}$  on the boundary of the element. In the case of  $\varepsilon = I$ ,  $(\varepsilon \mathbf{u} \cdot \mathbf{n}) \mathbf{n}$  is clearly the normal component of the vector  $\mathbf{u}$ . But for general  $\varepsilon$ ,  $(\varepsilon \mathbf{u} \cdot \mathbf{n}) \mathbf{n} + \mathbf{n} \times (\mathbf{u} \times \mathbf{n})$  is not a decomposition of the vector  $\mathbf{u}$  on  $\partial T$ .

Denote by  $\mathcal{Q}_h$  and  $\mathbf{Q}_h$  the  $L^2$  projections onto  $P_{k-1}(T)$  and  $[P_{k-1}(T)]^d$ , respectively.

**LEMMA 6.1.** [11, 15, 20] *The  $L^2$  projection operators  $Q_h$ ,  $\mathbf{Q}_h$ , and  $\mathcal{Q}_h$  satisfy the following commutative identities:*

$$(6.3) \quad \nabla_w \cdot (\varepsilon Q_h \mathbf{v}) = \mathcal{Q}_h(\nabla \cdot (\varepsilon \mathbf{v})), \quad \mathbf{v} \in H(\text{div}_\varepsilon; \Omega),$$

$$(6.4) \quad \nabla_w \cdot (\mu Q_h \mathbf{v}) = \mathcal{Q}_h(\nabla \cdot (\mu \mathbf{v})), \quad \mathbf{v} \in H(\text{div}_\mu; \Omega),$$

$$(6.5) \quad \nabla_w \times (Q_h \mathbf{v}) = \mathbf{Q}_h(\nabla \times \mathbf{v}), \quad \mathbf{v} \in H(\text{curl}; \Omega).$$

Let  $(\mathbf{u}_h; p_h) = (\{\mathbf{u}_0, \mathbf{u}_b\}; p_h)$  be the WG finite element solution arising from either the weak Galerkin Algorithm (4.3)-(4.4) or (4.5)-(4.6), and  $(\mathbf{u}; p)$  be the solution of the continuous model problem (2.3) or (2.4). Here the variable  $\mathbf{u}$  represents either the electric field intensity  $\mathbf{E}$  or the magnetic field intensity  $\mathbf{H}$ . The corresponding error functions are given as follows

$$(6.6) \quad \mathbf{e}_h = \{\mathbf{e}_0, \mathbf{e}_b\} = \{Q_0 \mathbf{u} - \mathbf{u}_0, Q_b \mathbf{u} - \mathbf{u}_b\},$$

$$(6.7) \quad \epsilon_h = Q_h p - p_h.$$

LEMMA 6.2. Assume that  $(\mathbf{w}; \rho) \in (H_0(\text{curl}; \Omega) \cap H(\text{div}_\varepsilon; \Omega)) \times L^2(\Omega)$  is sufficiently smooth on each element  $T \in \mathcal{T}_h$  and satisfies

$$(6.8) \quad \nabla \times (\mu^{-1} \nabla \times \mathbf{w}) + (i\omega\sigma - \omega^2\varepsilon)\mathbf{w} + \varepsilon \nabla \rho = \eta, \quad \text{in } \Omega,$$

$$(6.9) \quad \rho = 0, \quad \text{on } \Gamma.$$

Then, the following identity holds true:

$$(6.10) \quad \sum_{T \in \mathcal{T}_h} (\mu^{-1} \nabla_w \times (Q_h \mathbf{w}), \nabla_w \times \mathbf{v})_T + ((i\omega\sigma - \omega^2\varepsilon)Q_0 \mathbf{w}, \mathbf{v}_0)_T \\ - (\nabla_w \cdot (\varepsilon \mathbf{v}), Q_h \rho)_T = (\eta, \mathbf{v}_0) + l_w(\mathbf{v}) - \theta_\rho(\mathbf{v}),$$

for all  $\mathbf{v} \in \mathbf{V}_{h,0}^1$ . Here  $l_w(\mathbf{v})$  and  $\theta_\rho(\mathbf{v})$  are two functionals in the linear space  $\mathbf{V}_{h,0}^1$  given by

$$(6.11) \quad l_w(\mathbf{v}) = \sum_{T \in \mathcal{T}_h} \langle (Q_h - I)(\mu^{-1} \nabla \times \mathbf{w}), (\mathbf{v}_0 - \mathbf{v}_b) \times \mathbf{n} \rangle_{\partial T},$$

$$(6.12) \quad \theta_\rho(\mathbf{v}) = \sum_{T \in \mathcal{T}_h} \langle \rho - Q_h \rho, (\varepsilon \mathbf{v}_0 - \mathbf{v}_b) \cdot \mathbf{n} \rangle_{\partial T}.$$

*Proof.* Recall that  $\mu, \sigma, \omega$  and  $\varepsilon$  are assumed to be piecewise constants on the domain  $\Omega$  with respect to the given finite element partition. Thus, from (3.5) with  $\varphi = \mu^{-1} \nabla_w \times (Q_h \mathbf{w})$  we have

$$(\nabla_w \times \mathbf{v}, \mu^{-1} \nabla_w \times (Q_h \mathbf{w}))_T = \\ (\nabla \times \mathbf{v}_0, \mu^{-1} \nabla_w \times (Q_0 \mathbf{w}))_T - \langle (\mathbf{v}_b - \mathbf{v}_0) \times \mathbf{n}, \mu^{-1} \nabla_w \times (Q_h \mathbf{w}) \rangle_{\partial T}.$$

Using (6.5), the above equation can be rewritten as

$$(\mu^{-1} \nabla_w \times (Q_h \mathbf{w}), \nabla_w \times \mathbf{v})_T = \\ (\mu^{-1} \nabla \times \mathbf{w}, \nabla \times \mathbf{v}_0)_T + \langle \mathbf{Q}_h(\mu^{-1} \nabla \times \mathbf{w}), (\mathbf{v}_0 - \mathbf{v}_b) \times \mathbf{n} \rangle_{\partial T}.$$

Applying the integration by parts to the first term on the right-hand side yields

$$(6.13) \quad (\mu^{-1} \nabla_w \times (Q_h \mathbf{w}), \nabla_w \times \mathbf{v})_T + ((i\omega\sigma - \omega^2\varepsilon)Q_0 \mathbf{w}, \mathbf{v}_0)_T \\ = (\nabla \times (\mu^{-1} \nabla \times \mathbf{w}), \mathbf{v}_0)_T - \langle \mu^{-1} \nabla \times \mathbf{w}, \mathbf{v}_0 \times \mathbf{n} \rangle_{\partial T} \\ + \langle \mathbf{Q}_h(\mu^{-1} \nabla \times \mathbf{w}), (\mathbf{v}_0 - \mathbf{v}_b) \times \mathbf{n} \rangle_{\partial T} + ((i\omega\sigma - \omega^2\varepsilon)\mathbf{w}, \mathbf{v}_0)_T \\ = (\nabla \times (\mu^{-1} \nabla \times \mathbf{w}), \mathbf{v}_0)_T - \langle \mu^{-1} \nabla \times \mathbf{w}, \mathbf{v}_b \times \mathbf{n} \rangle_{\partial T} \\ + \langle (\mathbf{Q}_h - I)(\mu^{-1} \nabla \times \mathbf{w}), (\mathbf{v}_0 - \mathbf{v}_b) \times \mathbf{n} \rangle_{\partial T} + ((i\omega\sigma - \omega^2\varepsilon)\mathbf{w}, \mathbf{v}_0)_T.$$

Using (3.3) with  $\varphi = \mathcal{Q}_h \rho$  and the usual integration by parts, we obtain

$$\begin{aligned}
(6.14) \quad & (\nabla_w \cdot (\varepsilon \mathbf{v}), \mathcal{Q}_h \rho)_T \\
&= (\nabla \cdot (\varepsilon \mathbf{v}_0), \mathcal{Q}_h \rho)_T + \langle (\mathbf{v}_b - \varepsilon \mathbf{v}_0) \cdot \mathbf{n}, \mathcal{Q}_h \rho \rangle_{\partial T} \\
&= (\nabla \cdot (\varepsilon \mathbf{v}_0), \rho)_T + \langle (\mathbf{v}_b - \varepsilon \mathbf{v}_0) \cdot \mathbf{n}, \mathcal{Q}_h \rho \rangle_{\partial T} \\
&= -(\varepsilon \mathbf{v}_0, \nabla \rho)_T + \langle \varepsilon \mathbf{v}_0 \cdot \mathbf{n}, \rho \rangle_{\partial T} + \langle (\mathbf{v}_b - \varepsilon \mathbf{v}_0) \cdot \mathbf{n}, \mathcal{Q}_h \rho \rangle_{\partial T} \\
&= -(\mathbf{v}_0, \varepsilon \nabla \rho)_T + \langle (\mathbf{v}_b - \varepsilon \mathbf{v}_0) \cdot \mathbf{n}, \mathcal{Q}_h \rho - \rho \rangle_{\partial T} + \langle \mathbf{v}_b \cdot \mathbf{n}, \rho \rangle_{\partial T}.
\end{aligned}$$

Summing (6.13) over all the elements  $T \in \mathcal{T}_h$  yields

$$\begin{aligned}
(6.15) \quad & \sum_{T \in \mathcal{T}_h} (\mu^{-1} \nabla_w \times (Q_h \mathbf{w}), \nabla_w \times \mathbf{v})_T + ((i\omega\sigma - \omega^2\varepsilon)Q_0 \mathbf{w}, \mathbf{v}_0)_T \\
&= \sum_{T \in \mathcal{T}_h} (\nabla \times (\mu^{-1} \nabla \times \mathbf{w}), \mathbf{v}_0)_T + \langle (\mathbf{Q}_h - I)(\mu^{-1} \nabla \times \mathbf{w}), (\mathbf{v}_0 - \mathbf{v}_b) \times \mathbf{n} \rangle_{\partial T} \\
&\quad + ((i\omega\sigma - \omega^2\varepsilon)\mathbf{w}, \mathbf{v}_0)_T,
\end{aligned}$$

where we have used two properties: (1) the cancelation property for the boundary integrals on interior edges/faces, and (2) the fact that  $\mathbf{v}_b \times \mathbf{n} = 0$  on  $\Gamma$ . Similarly, summing (6.14) over all the elements  $T \in \mathcal{T}_h$  leads to

$$\begin{aligned}
(6.16) \quad & \sum_{T \in \mathcal{T}_h} (\nabla_w \cdot (\varepsilon \mathbf{v}), \mathcal{Q}_h \rho)_T = -(\mathbf{v}_0, \varepsilon \nabla \rho) + \sum_{T \in \mathcal{T}_h} \langle (\varepsilon \mathbf{v}_0 - \mathbf{v}_b) \cdot \mathbf{n}, \rho - \mathcal{Q}_h \rho \rangle_{\partial T} \\
&\quad + \sum_{e \in \mathcal{E}_h \cap \Gamma} \langle \mathbf{v}_b \cdot \mathbf{n}, \rho \rangle_e.
\end{aligned}$$

The third term on the right-hand side of (6.16) vanishes if  $\rho$  satisfies the boundary condition (6.9). Thus, the equation (6.10) holds true from (6.15) and (6.16). This completes the proof of the lemma.  $\square$

LEMMA 6.3. Assume that  $(\mathbf{w}; \rho) \in (H_0(\operatorname{div}_\mu; \Omega) \cap H(\operatorname{curl}; \Omega)) \times L_0^2(\Omega)$  is sufficiently smooth on each element  $T \in \mathcal{T}_h$  and satisfies

$$(6.17) \quad \nabla \times ((i\omega\varepsilon + \sigma)^{-1} \nabla \times \mathbf{w}) + i\omega\mu\mathbf{w} + \mu\nabla\rho = \eta, \quad \text{in } \Omega,$$

$$(6.18) \quad (i\omega\varepsilon + \sigma)^{-1} \nabla \times \mathbf{w} \times \mathbf{n} = 0, \quad \text{on } \Gamma.$$

Then, we have the following identity:

$$\begin{aligned}
(6.19) \quad & \sum_{T \in \mathcal{T}_h} ((i\omega\varepsilon + \sigma)^{-1} \nabla_w \times (Q_h \mathbf{w}), \nabla_w \times \mathbf{v})_T + (i\omega\mu Q_0 \mathbf{w}, \mathbf{v}_0)_T \\
&\quad - (\nabla_w \cdot (\mu \mathbf{v}), \mathcal{Q}_h \rho)_T = (\eta, \mathbf{v}_0) + l'_w(\mathbf{v}) - \theta'_\rho(\mathbf{v}),
\end{aligned}$$

for all  $\mathbf{v} \in \mathbf{V}_{h,0}^2$ . Here  $l'_w(\mathbf{v})$  and  $\theta'_\rho(\mathbf{v})$  are two functionals in the linear space  $\mathbf{V}_{h,0}^2$  given by

$$(6.20) \quad l'_w(\mathbf{v}) = \sum_{T \in \mathcal{T}_h} \langle (\mathbf{Q}_h - I)(i\omega\varepsilon + \sigma)^{-1} \nabla \times \mathbf{w}, (\mathbf{v}_0 - \mathbf{v}_b) \times \mathbf{n} \rangle_{\partial T},$$

$$(6.21) \quad \theta'_\rho(\mathbf{v}) = \sum_{T \in \mathcal{T}_h} \langle \rho - \mathcal{Q}_h \rho, (\mu \mathbf{v}_0 - \mathbf{v}_b) \cdot \mathbf{n} \rangle_{\partial T}.$$

*Proof.* Since  $\mu$ ,  $\sigma$ ,  $\omega$  and  $\varepsilon$  are piecewise constants on the domain  $\Omega$  with respect to the given finite element partitions, then from (3.5) with  $\varphi = (i\omega\varepsilon + \sigma)^{-1}\nabla_w \times (Q_h \mathbf{w})$  we obtain

$$\begin{aligned} (\nabla_w \times \mathbf{v}, (i\omega\varepsilon + \sigma)^{-1}\nabla_w \times (Q_h \mathbf{w}))_T &= (\nabla \times \mathbf{v}_0, (i\omega\varepsilon + \sigma)^{-1}\nabla_w \times (Q_h \mathbf{w}))_T \\ &\quad - \langle (\mathbf{v}_b - \mathbf{v}_0) \times \mathbf{n}, (i\omega\varepsilon + \sigma)^{-1}\nabla_w \times (Q_h \mathbf{w}) \rangle_{\partial T}, \end{aligned}$$

which, combined with (6.5), gives rise to

$$\begin{aligned} ((i\omega\varepsilon + \sigma)^{-1}\nabla_w \times (Q_h \mathbf{w}), \nabla_w \times \mathbf{v})_T &= \\ ((i\omega\varepsilon + \sigma)^{-1}\nabla \times \mathbf{w}, \nabla \times \mathbf{v}_0)_T &+ \langle \mathbf{Q}_h((i\omega\varepsilon + \sigma)^{-1}\nabla \times \mathbf{w}), (\mathbf{v}_0 - \mathbf{v}_b) \times \mathbf{n} \rangle_{\partial T}. \end{aligned}$$

Now applying the integration by parts to the first term on the right-hand side of the above identity yields

$$\begin{aligned} &((i\omega\varepsilon + \sigma)^{-1}\nabla_w \times (Q_h \mathbf{w}), \nabla_w \times \mathbf{v})_T + (i\omega\mu Q_0 \mathbf{w}, \mathbf{v}_0)_T \\ &= (\nabla \times ((i\omega\varepsilon + \sigma)^{-1}\nabla \times \mathbf{w}), \mathbf{v}_0)_T - \langle (i\omega\varepsilon + \sigma)^{-1}\nabla \times \mathbf{w}, \mathbf{v}_0 \times \mathbf{n} \rangle_{\partial T} \\ (6.22) \quad &+ \langle \mathbf{Q}_h((i\omega\varepsilon + \sigma)^{-1}\nabla \times \mathbf{w}), (\mathbf{v}_0 - \mathbf{v}_b) \times \mathbf{n} \rangle_{\partial T} + (i\omega\mu \mathbf{w}, \mathbf{v}_0)_T \\ &= (\nabla \times ((i\omega\varepsilon + \sigma)^{-1}\nabla \times \mathbf{w}), \mathbf{v}_0)_T - \langle (i\omega\varepsilon + \sigma)^{-1}\nabla \times \mathbf{w}, \mathbf{v}_b \times \mathbf{n} \rangle_{\partial T} \\ &+ \langle (\mathbf{Q}_h - I)((i\omega\varepsilon + \sigma)^{-1}\nabla \times \mathbf{w}), (\mathbf{v}_0 - \mathbf{v}_b) \times \mathbf{n} \rangle_{\partial T} + (i\omega\mu \mathbf{w}, \mathbf{v}_0)_T. \end{aligned}$$

Using (3.3) with  $\varphi = Q_h \rho$  and the usual integration by parts, we obtain

$$\begin{aligned} &(\nabla_w \cdot (\mu \mathbf{v}), Q_h \rho)_T \\ &= (\nabla \cdot (\mu \mathbf{v}_0), Q_h \rho)_T + \langle (\mathbf{v}_b - \mu \mathbf{v}_0) \cdot \mathbf{n}, Q_h \rho \rangle_{\partial T} \\ (6.23) \quad &= (\nabla \cdot (\mu \mathbf{v}_0), \rho)_T + \langle (\mathbf{v}_b - \mu \mathbf{v}_0) \cdot \mathbf{n}, Q_h \rho \rangle_{\partial T} \\ &= -(\mu \mathbf{v}_0, \nabla \rho)_T + \langle \mu \mathbf{v}_0 \cdot \mathbf{n}, \rho \rangle_{\partial T} + \langle (\mathbf{v}_b - \mu \mathbf{v}_0) \cdot \mathbf{n}, Q_h \rho \rangle_{\partial T} \\ &= -(\mathbf{v}_0, \mu \nabla \rho)_T + \langle (\mathbf{v}_b - \mu \mathbf{v}_0) \cdot \mathbf{n}, Q_h \rho - \rho \rangle_{\partial T} + \langle \mathbf{v}_b \cdot \mathbf{n}, \rho \rangle_{\partial T}. \end{aligned}$$

Summing (6.22) over all the elements  $T \in \mathcal{T}_h$  yields

$$\begin{aligned} &\sum_{T \in \mathcal{T}_h} ((i\omega\varepsilon + \sigma)^{-1}\nabla_w \times (Q_h \mathbf{w}), \nabla_w \times \mathbf{v})_T + (i\omega\mu Q_0 \mathbf{w}, \mathbf{v}_0)_T \\ (6.24) \quad &= \sum_{T \in \mathcal{T}_h} (\nabla \times ((i\omega\varepsilon + \sigma)^{-1}\nabla \times \mathbf{w}), \mathbf{v}_0)_T \\ &+ \langle (\mathbf{Q}_h - I)((i\omega\varepsilon + \sigma)^{-1}\nabla \times \mathbf{w}), (\mathbf{v}_0 - \mathbf{v}_b) \times \mathbf{n} \rangle_{\partial T} + (i\omega\mu \mathbf{w}, \mathbf{v}_0)_T, \end{aligned}$$

where we have used two properties: the first is the cancelation property for the boundary integrals on interior edges/faces, and the second is the boundary condition (6.18). Similarly, summing (6.23) over all the elements  $T \in \mathcal{T}_h$ , we obtain

$$\begin{aligned} &\sum_{T \in \mathcal{T}_h} (\nabla_w \cdot (\mu \mathbf{v}), Q_h \rho)_T = -(\mathbf{v}_0, \mu \nabla \rho) + \sum_{T \in \mathcal{T}_h} \langle (\mu \mathbf{v}_0 - \mathbf{v}_b) \cdot \mathbf{n}, \rho - Q_h \rho \rangle_{\partial T} \\ (6.25) \quad &+ \sum_{e \in \mathcal{E}_h \cap \Gamma} \langle \mathbf{v}_b \cdot \mathbf{n}, \rho \rangle_e. \end{aligned}$$

The third term on the right-hand side of (6.25) vanishes for  $\mathbf{v} \in \mathbf{V}_{h,0}^2$ . Thus, the equation (6.19) holds true from (6.24) and (6.25). This completes the proof of the lemma.  $\square$

**THEOREM 6.4.** *Let  $(\mathbf{u}; p)$  be the solution of the problem (2.3) for the electric field and  $(\mathbf{u}_h; p_h)$  be its numerical solution arising from the WG finite element scheme*

(4.3)-(4.4). Define the error functions  $\mathbf{e}_h$  and  $\epsilon_h$  by (6.6)-(6.7). Then,  $\mathbf{e}_h \in \mathbf{V}_{h,0}^1$  and the following error equations hold true:

$$(6.26) \quad a_1(\mathbf{e}_h, \mathbf{v}) - b_1(\mathbf{v}, \epsilon_h) = \varphi_{\mathbf{u},p}(\mathbf{v}), \quad \forall \mathbf{v} \in \mathbf{V}_{h,0}^1,$$

$$(6.27) \quad b_1(\mathbf{e}_h, q) = 0, \quad \forall q \in W_h^1,$$

where

$$(6.28) \quad \varphi_{\mathbf{u},p}(\mathbf{v}) = l_{\mathbf{u}}(\mathbf{v}) - \theta_p(\mathbf{v}) + s_1(Q_h \mathbf{u}, \mathbf{v}).$$

*Proof.* Let  $(\mathbf{u}; p)$  be the solution of the model problem (2.3). It is not hard to see that the following holds true:

$$\begin{aligned} \nabla \times (\mu^{-1} \nabla \times \mathbf{u}) + (i\omega\sigma - \omega^2 \varepsilon) \mathbf{u} + \varepsilon \nabla p &= -i\omega \mathbf{j}_e, & \text{in } \Omega, \\ p &= 0, & \text{on } \Gamma. \end{aligned}$$

It follows from Lemma 6.2 that

$$\begin{aligned} & \sum_{T \in \mathcal{T}_h} (\mu^{-1} \nabla_w \times (Q_h \mathbf{u}), \nabla_w \times \mathbf{v})_T + ((i\omega\sigma - \omega^2 \varepsilon) Q_0 \mathbf{u}, \mathbf{v}_0)_T - (\nabla_w \cdot (\varepsilon \mathbf{v}), \mathcal{Q}_h p)_T \\ &= (-i\omega \mathbf{j}_e, \mathbf{v}_0) + l_{\mathbf{u}}(\mathbf{v}) - \theta_p(\mathbf{v}), \end{aligned}$$

for all  $\mathbf{v} \in \mathbf{V}_{h,0}^1$ , which gives

$$(6.29) \quad a_1(Q_h \mathbf{u}, \mathbf{v}) - b_1(\mathbf{v}, \mathcal{Q}_h p) = (-i\omega \mathbf{j}_e, \mathbf{v}_0) + l_{\mathbf{u}}(\mathbf{v}) - \theta_p(\mathbf{v}) + s_1(Q_h \mathbf{u}, \mathbf{v}).$$

Subtracting (4.3) from (6.29) gives rise to the first error equation (6.26).

Next, from the second equation in (2.3) and the commutative relation (6.3), we have for any  $q \in W_h^1$ ,

$$(6.30) \quad (\rho, q) = \sum_{T \in \mathcal{T}_h} (\nabla \cdot (\varepsilon \mathbf{u}), q)_T = \sum_{T \in \mathcal{T}_h} (\mathcal{Q}_h (\nabla \cdot (\varepsilon \mathbf{u})), q)_T = \sum_{T \in \mathcal{T}_h} (\nabla_w \cdot (\varepsilon Q_h \mathbf{u}), q)_T.$$

The difference of (6.30) and (4.4) yields the second error equation (6.27). This completes the proof.  $\square$

**THEOREM 6.5.** *Let  $(\mathbf{u}; p)$  be the solution of the problem (2.4) for the magnetic field and  $(\mathbf{u}_h; p_h)$  be its numerical solution arising from the WG finite element scheme (4.5)-(4.6). Denote the error functions  $\mathbf{e}_h$  and  $\epsilon_h$  by (6.6)-(6.7). Then,  $\mathbf{e}_h \in \mathbf{V}_{h,0}^2$  and the following error equations hold true:*

$$(6.31) \quad a_2(\mathbf{e}_h, \mathbf{v}) - b_2(\mathbf{v}, \epsilon_h) = \varphi'_{\mathbf{u},p}(\mathbf{v}), \quad \forall \mathbf{v} \in \mathbf{V}_{h,0}^2,$$

$$(6.32) \quad b_2(\mathbf{e}_h, q) = 0, \quad \forall q \in W_h^2,$$

where

$$(6.33) \quad \varphi'_{\mathbf{u},p}(\mathbf{v}) = l'_{\mathbf{u}}(\mathbf{v}) - \theta'_p(\mathbf{v}) + s_2(Q_h \mathbf{u}, \mathbf{v}).$$



*Proof.* Let  $(\mathbf{u}; p)$  be the solution of the model problem (2.4). It is not hard to see that the following holds true:

$$\begin{aligned} \nabla \times ((i\omega\varepsilon + \sigma)^{-1} \nabla \times \mathbf{u}) + i\omega\mu\mathbf{u} + \mu\nabla p &= \nabla \times (i\omega\varepsilon + \sigma)^{-1} \mathbf{j}_e, & \text{in } \Omega, \\ (i\omega\varepsilon + \sigma)^{-1} \nabla \times \mathbf{u} \times \mathbf{n} &= 0, & \text{on } \Gamma. \end{aligned}$$

It follows from Lemma 6.3 that

$$\begin{aligned} &\sum_{T \in \mathcal{T}_h} ((i\omega\varepsilon + \sigma)^{-1} \nabla_w \times (Q_h \mathbf{u}), \nabla_w \times \mathbf{v})_T + (i\omega\mu Q_0 \mathbf{u}, \mathbf{v}_0)_T - (\nabla_w \cdot (\mu \mathbf{v}), Q_h p)_T \\ &= (\nabla \times (i\omega\varepsilon + \sigma)^{-1} \mathbf{j}_e, \mathbf{v}_0) + l'_u(\mathbf{v}) - \theta'_p(\mathbf{v}), \end{aligned}$$

for all  $\mathbf{v} \in \mathbf{V}_{h,0}^2$ , which gives

$$(6.34) \quad a_2(Q_h \mathbf{u}, \mathbf{v}) - b_2(\mathbf{v}, Q_h p) = (\nabla \times (i\omega\varepsilon + \sigma)^{-1} \mathbf{j}_e, \mathbf{v}_0) + l'_u(\mathbf{v}) - \theta'_p(\mathbf{v}) + s_2(Q_h \mathbf{u}, \mathbf{v}).$$

Subtracting (4.5) from (6.34) gives rise to the first error equation (6.31).

Next, from the second equation in (2.4) and the commutative relation (6.4), we have for any  $q \in W_h^2$ ,

$$(6.35) \quad 0 = \sum_{T \in \mathcal{T}_h} (\nabla \cdot (\mu \mathbf{u}), q)_T = \sum_{T \in \mathcal{T}_h} (Q_h(\nabla \cdot (\mu \mathbf{u})), q)_T = \sum_{T \in \mathcal{T}_h} (\nabla_w \cdot (\mu Q_h \mathbf{u}), q)_T.$$

The difference of (6.35) and (4.6) yields the second error equation (6.32). This completes the proof.  $\square$

**7. Error Analysis.** The goal of this section is to derive some error estimates for the numerical approximations  $\mathbf{E}_h$  and  $\mathbf{H}_h$  arising from the weak Galerkin algorithms 1-2 for the time-harmonic Maxwell equations. Recall that the error functions, denoted by  $\mathbf{e}_h$  and  $\epsilon_h$ , are defined as the difference of the numerical approximation and the  $L^2$  projection of the exact solution. The error equations as presented in Theorems 6.4-6.5 play an important role in the convergence analysis.

**7.1. Some technical inequalities.** Assume that the finite element partition  $\mathcal{T}_h$  of  $\Omega$  is shape regular in the sense as detailed in [19]. Let  $T \in \mathcal{T}_h$  be an element with  $e$  as an edge/face. It is known that the following trace inequality holds true

$$(7.1) \quad \|\psi\|_e^2 \lesssim (h_T^{-1} \|\psi\|_T^2 + h_T \|\nabla \psi\|_T^2), \quad \forall \psi \in H^1(T).$$

For polynomial functions, we have the following inverse inequality

$$(7.2) \quad \|\nabla \phi\|_T \lesssim h_T^{-1} \|\phi\|_T.$$

In particular, by combining (7.1) with (7.2), we arrive at

$$(7.3) \quad \|\phi\|_e^2 \lesssim h_T^{-1} \|\phi\|_T^2$$

for any polynomial  $\phi$  on  $T$  with degree no more than a prescribed number.

LEMMA 7.1. [19] Let  $k \geq 1$  be the order of the WG finite elements, and  $1 \leq r \leq k$ . Let  $\mathbf{w} \in [H^{r+1}(\Omega)]^d$ ,  $\rho \in H^r(\Omega)$ , and  $0 \leq m \leq 1$ . There holds

$$(7.4) \quad \sum_{T \in \mathcal{T}_h} h_T^{2m} \|\mathbf{w} - Q_0 \mathbf{w}\|_{T,m}^2 \lesssim h^{2(r+1)} \|\mathbf{w}\|_{r+1}^2,$$

$$(7.5) \quad \sum_{T \in \mathcal{T}_h} h_T^{2m} \|\nabla \times \mathbf{w} - \mathbf{Q}_h(\nabla \times \mathbf{w})\|_{T,m}^2 \lesssim h^{2r} \|\mathbf{w}\|_{r+1}^2,$$

$$(7.6) \quad \sum_{T \in \mathcal{T}_h} h_T^{2m} \|\rho - \mathcal{Q}_h \rho\|_{T,m}^2 \lesssim h^{2r} \|\rho\|_r^2.$$

For convenience, we introduce two semi-norms in the WG finite element space  $\mathbf{V}_h$ ; i.e.,

$$|\mathbf{v}|_{1,h} = \left( \sum_{T \in \mathcal{T}_h} h_T^{-1} \|(\mathbf{v}_0 - \mathbf{v}_b) \times \mathbf{n}\|_{\partial T}^2 + h_T^{-1} \|(\varepsilon \mathbf{v}_0 - \mathbf{v}_b) \cdot \mathbf{n}\|_{\partial T}^2 \right)^{\frac{1}{2}},$$

$$|\mathbf{v}|_{2,h} = \left( \sum_{T \in \mathcal{T}_h} h_T^{-1} \|(\mathbf{v}_0 - \mathbf{v}_b) \times \mathbf{n}\|_{\partial T}^2 + h_T^{-1} \|(\mu \mathbf{v}_0 - \mathbf{v}_b) \cdot \mathbf{n}\|_{\partial T}^2 \right)^{\frac{1}{2}}.$$

LEMMA 7.2. [15] Assume that the finite element partition  $\mathcal{T}_h$  of  $\Omega$  is shape regular and  $1 \leq r \leq k$ . Let  $\mathbf{w} \in [H^{r+1}(\Omega)]^d$  and  $p \in H^r(\Omega)$ . Then, we have

$$\begin{aligned} |s_1(Q_h \mathbf{w}, \mathbf{v})| &\lesssim h^r \|\mathbf{w}\|_{r+1} |\mathbf{v}|_{1,h}, \\ |s_2(Q_h \mathbf{w}, \mathbf{v})| &\lesssim h^r \|\mathbf{w}\|_{r+1} |\mathbf{v}|_{2,h}, \\ |l_{\mathbf{w}}(\mathbf{v})| &\lesssim h^r \|\mathbf{w}\|_{r+1} |\mathbf{v}|_{1,h}, \\ |l'_{\mathbf{w}}(\mathbf{v})| &\lesssim h^r \|\mathbf{w}\|_{r+1} |\mathbf{v}|_{2,h}, \\ |\theta_p(\mathbf{v})| &\lesssim h^r \|p\|_r |\mathbf{v}|_{1,h}, \\ |\theta'_p(\mathbf{v})| &\lesssim h^r \|p\|_r |\mathbf{v}|_{2,h}, \end{aligned}$$

for any  $\mathbf{v} \in \mathbf{V}_h$ . Here,  $l_{\mathbf{w}}(\cdot)$ ,  $\theta_p(\cdot)$  and  $l'_{\mathbf{w}}(\cdot)$ ,  $\theta'_p(\cdot)$  are defined in (6.11)-(6.12) and (6.20)-(6.21), respectively.

**7.2. Error estimates.** We are now in a position to present some error estimates for the weak Galerkin algorithms 1-2.

THEOREM 7.3. Assume that  $k \geq 1$  is the order of the WG finite elements for (4.3)-(4.4). Let  $(\mathbf{E}; p) \in [H^{k+1}(\Omega)]^d \times H^k(\Omega)$  be the solution of the problem (2.3) and  $(\mathbf{E}_h; p_h) \in \mathbf{V}_{h,0}^1 \times W_h^1$  be the WG finite element solution arising from (4.3)-(4.4). Then, we have the following estimate

$$(7.7) \quad \|Q_h \mathbf{E} - \mathbf{E}_h\|_{\mathbf{V}_{h,0}^1} + \|\mathcal{Q}_h p - p_h\|_{W_h^1} \lesssim h^k (\|\mathbf{E}\|_{k+1} + \|p\|_k).$$

*Proof.* Theorem 6.4 implies that the error functions  $\mathbf{e}_h = Q_h \mathbf{E} - \mathbf{E}_h$  and  $\epsilon_h = \mathcal{Q}_h p - p_h$  satisfy the error equations (6.26)-(6.27). By letting  $\mathbf{v} = \mathbf{e}_h$  in (6.26) and then using (6.27) we obtain

$$(7.8) \quad a_1(\mathbf{e}_h, \mathbf{e}_h) = \varphi_{\mathbf{E},p}(\mathbf{e}_h).$$

The right-hand side of (7.8) can be estimated by using Lemma 7.2 as follows

$$|\varphi_{\mathbf{E},p}(\mathbf{e}_h)| \lesssim h^k(\|\mathbf{E}\|_{k+1} + \|p\|_k)|\mathbf{e}_h|_{1,h}.$$

Substituting the above into (7.8) yields

$$|a_1(\mathbf{e}_h, \mathbf{e}_h)| \lesssim h^k(\|\mathbf{E}\|_{k+1} + \|p\|_k)|\mathbf{e}_h|_{1,h},$$

which, together with the coercivity  $|\mathbf{e}_h|_{1,h}^2 \lesssim |a_1(\mathbf{e}_h, \mathbf{e}_h)|$ , leads to

$$(7.9) \quad |a_1(\mathbf{e}_h, \mathbf{e}_h)|^{1/2} \lesssim h^k(\|\mathbf{E}\|_{k+1} + \|p\|_k).$$

From the imaginary part of  $a_1(\mathbf{e}_h, \mathbf{e}_h)$  and (7.9), we have

$$\left( \sum_{T \in \mathcal{T}_h} \|\mathbf{e}_0\|_T^2 \right)^{1/2} \lesssim h^k(\|\mathbf{E}\|_{k+1} + \|p\|_k),$$

which, combining with the real part of  $a_1(\mathbf{e}_h, \mathbf{e}_h)$  and (7.9), yields

$$\begin{aligned} & \left( \sum_{T \in \mathcal{T}_h} \|\nabla_w \times \mathbf{e}_h\|_T^2 + h_T^{-1} \|(\varepsilon \mathbf{e}_0 - \mathbf{e}_b) \cdot \mathbf{n}\|_{\partial T}^2 + h_T^{-1} \|(\mathbf{e}_0 - \mathbf{e}_b) \times \mathbf{n}\|_{\partial T}^2 \right)^{1/2} \\ & \lesssim h^k(\|\mathbf{E}\|_{k+1} + \|p\|_k). \end{aligned}$$

Thus, we have

$$\|\mathbf{e}_h\|_{\mathbf{V}_{h,0}^1} \lesssim h^k(\|\mathbf{E}\|_{k+1} + \|p\|_k).$$

The error function  $\epsilon_h$  can be estimated by using the *inf-sup* condition derived in Lemma 5.3. To this end, from the equation (6.26), we have

$$(7.10) \quad b_1(\mathbf{v}, \epsilon_h) = -\varphi_{\mathbf{E},p}(\mathbf{v}) + a_1(\mathbf{e}_h, \mathbf{v}).$$

By using Lemma 5.3 and letting  $\mathbf{v} = \mathbf{v}_{\epsilon_h}$  in (7.10) we arrive at

$$\|\epsilon_h\|_{W_h^1}^2 \lesssim |\varphi_{\mathbf{E},p}(\mathbf{v}_{\epsilon_h})| + |a_1(\mathbf{e}_h, \mathbf{v}_{\epsilon_h})|.$$

It now follows from Lemma 7.2 and the error estimate (7.9) that

$$\|\epsilon_h\|_{W_h^1}^2 \lesssim h^k(\|\mathbf{E}\|_{k+1} + \|p\|_k) \|\mathbf{v}_{\epsilon_h}\|_{\mathbf{V}_{h,0}^1},$$

which, together with (5.14), leads to

$$\|\epsilon_h\|_{W_h^1} \lesssim h^k(\|\mathbf{E}\|_{k+1} + \|p\|_k).$$

This completes the proof of the theorem.  $\square$

**THEOREM 7.4.** *Assume that  $k \geq 1$  is the order of the WG finite elements employed in the scheme (4.5)-(4.6). Let  $(\mathbf{H}; p) \in [H^{k+1}(\Omega)]^d \times H^k(\Omega)$  be the solution of the problem (2.4) and  $(\mathbf{H}_h; p_h) \in \mathbf{V}_{h,0}^2 \times W_h^2$  be the WG finite element solution arising from (4.5)-(4.6). Then, we have*

$$(7.11) \quad \|Q_h \mathbf{H} - \mathbf{H}_h\|_{\mathbf{V}_{h,0}^2} + \|\mathcal{Q}_h p - p_h\|_{W_h^2} \lesssim h^k(\|\mathbf{H}\|_{k+1} + \|p\|_k).$$

*Proof.* From Theorem 6.5 we see that the error functions  $\mathbf{e}_h = Q_h \mathbf{H} - \mathbf{H}_h$  and  $\epsilon_h = \mathcal{Q}_h p - p_h$  satisfy the error equations (6.31)-(6.32). By setting  $\mathbf{v} = \mathbf{e}_h$  in (6.31) and then using (6.32) we obtain

$$(7.12) \quad a_2(\mathbf{e}_h, \mathbf{e}_h) = \varphi'_{\mathbf{H},p}(\mathbf{e}_h).$$

The right-hand side of (7.12) can be handled by using Lemma 7.2 as follows

$$|\varphi'_{\mathbf{H},p}(\mathbf{e}_h)| \lesssim h^k(\|\mathbf{H}\|_{k+1} + \|p\|_k) |\mathbf{e}_h|_{2,h}.$$

Substituting the above estimate into (7.12) yields

$$|a_2(\mathbf{e}_h, \mathbf{e}_h)| \lesssim h^k(\|\mathbf{H}\|_{k+1} + \|p\|_k) |\mathbf{e}_h|_{2,h},$$

which, together with the coercivity estimate  $|\mathbf{e}_h|_{2,h}^2 \lesssim |a_2(\mathbf{e}_h, \mathbf{e}_h)|$ , leads to

$$(7.13) \quad |a_2(\mathbf{e}_h, \mathbf{e}_h)|^{1/2} \lesssim h^k(\|\mathbf{H}\|_{k+1} + \|p\|_k),$$

From the real part of  $a_2(\mathbf{e}_h, \mathbf{e}_h)$  and (7.13), we obtain

$$(7.14) \quad \begin{aligned} & \left( \sum_{T \in \mathcal{T}_h} \|\nabla_w \times \mathbf{e}_h\|_T^2 \right)^{1/2} \lesssim h^k(\|\mathbf{H}\|_{k+1} + \|p\|_k), \\ & \left( \sum_{T \in \mathcal{T}_h} h_T^{-1} \|(\mu \mathbf{e}_0 - \mathbf{e}_b) \cdot \mathbf{n}\|_{\partial T}^2 \right)^{1/2} \lesssim h^k(\|\mathbf{H}\|_{k+1} + \|p\|_k), \\ & \left( \sum_{T \in \mathcal{T}_h} h_T^{-1} \|(\mathbf{e}_0 - \mathbf{e}_b) \times \mathbf{n}\|_{\partial T}^2 \right)^{1/2} \lesssim h^k(\|\mathbf{H}\|_{k+1} + \|p\|_k). \end{aligned}$$

Combining with the imaginary part of  $a_2(\mathbf{e}_h, \mathbf{e}_h)$ , (7.13) and (7.14) gives rise to

$$\left( \sum_{T \in \mathcal{T}_h} \|\mathbf{e}_0\|_T^2 \right)^{1/2} \lesssim h^k(\|\mathbf{H}\|_{k+1} + \|p\|_k).$$

Thus,

$$\|\mathbf{e}_h\|_{\mathbf{V}_{h,0}^2} \lesssim h^k(\|\mathbf{H}\|_{k+1} + \|p\|_k).$$

The error function  $\epsilon_h$  can be estimated by using the *inf-sup* condition derived in Lemma 5.4. To this end, from the equation (6.31), we have

$$(7.15) \quad b_2(\mathbf{v}, \epsilon_h) = -\varphi'_{\mathbf{H},p}(\mathbf{v}) + a_2(\mathbf{e}_h, \mathbf{v}).$$

Using Lemma 5.4 and letting  $\mathbf{v} = \mathbf{v}_{\epsilon_h}$  in (7.15) yields

$$\|\epsilon_h\|_{W_h^2}^2 \lesssim |\varphi'_{\mathbf{H},p}(\mathbf{v}_{\epsilon_h})| + |a_2(\mathbf{e}_h, \mathbf{v}_{\epsilon_h})|.$$

It now follows from Lemma 7.2 and the error estimate (7.13) that

$$\|\epsilon_h\|_{W_h^2}^2 \lesssim h^k(\|\mathbf{H}\|_{k+1} + \|p\|_k) \|\mathbf{v}_{\epsilon_h}\|_{\mathbf{V}_{h,0}^2},$$

which, together with (5.20), leads to

$$\|\epsilon_h\|_{W_h^2} \lesssim h^k(\|\mathbf{H}\|_{k+1} + \|p\|_k).$$

This completes the proof of the theorem.  $\square$

**7.3.  $L^2$ -error estimates.** In this subsection, we shall present a  $L^2$ -error estimate for the components  $\mathbf{e}_0$  and  $\mathbf{e}_b$  in the error function  $\mathbf{e}_h$  for the WG algorithms 1 and 2. To this end, let us introduce a  $L^2$ -like norm for the edge/face component  $\mathbf{v}_b$  in the weak function  $\mathbf{v} = \{\mathbf{v}_0; \mathbf{v}_b\} \in \mathbf{V}_h$  as follows:

$$\|\mathbf{v}_b\|_{\varepsilon_h} = \left( \sum_{T \in T_h} h_T \int_{\partial T} |\mathbf{v}_b|^2 ds \right)^{\frac{1}{2}}.$$

**THEOREM 7.5.** *Let  $k \geq 1$  be the order of the WG finite element employed in the scheme (4.3)-(4.4). Let  $(\mathbf{E}; p) \in [H^{k+1}(\Omega)]^d \times H^k(\Omega)$  and  $(\mathbf{E}_h; p_h) \in \mathbf{V}_{h,0}^1 \times W_h^1$  be the solutions of the problem (2.3) and (4.3)-(4.4), respectively. Then, the following estimate holds true:*

$$\|Q_0 \mathbf{E} - \mathbf{E}_0\| \lesssim h^{k+1} (\|\mathbf{E}\|_{k+1} + \|p\|_k),$$

$$\|Q_b \mathbf{E} - \mathbf{E}_b\|_{\varepsilon_h} \lesssim h^{k+1} (\|\mathbf{E}\|_{k+1} + \|p\|_k).$$

Likewise, for the magnetic field intensity approximation, we have the following result.

**THEOREM 7.6.** *Let  $k \geq 1$  be the order of the WG finite elements employed in the WG scheme (4.5)-(4.6). Let  $(\mathbf{H}; p) \in [H^{k+1}(\Omega)]^d \times H^k(\Omega)$  and  $(\mathbf{H}_h; p_h) \in \mathbf{V}_{h,0}^2 \times W_h^2$  be the solutions of the problem (2.4) and (4.5)-(4.6), respectively. Then the following  $L^2$ -error estimates hold true:*

$$\|Q_0 \mathbf{H} - \mathbf{H}_0\| \lesssim h^{k+1} (\|\mathbf{H}\|_{k+1} + \|p\|_k),$$

$$\|Q_b \mathbf{H} - \mathbf{H}_b\|_{\varepsilon_h} \lesssim h^{k+1} (\|\mathbf{H}\|_{k+1} + \|p\|_k).$$

A proof for Theorems 7.5 and 7.6 can be given by following a routine duality argument readily available in the finite element method. Readers are referred to [15] for more details on a model problem that resembles the time harmonic Maxwell equations.

#### REFERENCES

- [1] I. BABUŠKA, *The finite element method with Lagrange multipliers*, Numer. Math., Vol. 20, pp. 179-192, 1973.
- [2] S. BRENNER, F. LI AND L. SUNG, *A locally divergence-free interior penalty method for two dimensional curl-curl problems*, SIAM J. Numer. Anal., Vol. 42, pp. 1190-1211, 2008.
- [3] F. BREZZI, *On the existence, uniqueness, and approximation of saddle point problems arising from Lagrange multipliers*, RAIRO, Vol. 8, pp. 129-151, 1974.
- [4] W. CAI, *Computational methods for electromagnetic phenomena: electrostatics in solvation, scattering, and electron transport*. Cambridge University Press, New York, 2013.
- [5] P.G. CIARLET, *The Finite Element Method for Elliptic Problems*, North-Holland, 1978.
- [6] P. HOUSTON, I. PERUGIA AND D. SCHOTZAU, *Mixed discontinuous Galerkin approximation of the Maxwell operator*, Tech. report 02-16, University of Basel, Department of Mathematics, Basel, Switzerland, 2002.

- [7] P. HOUSTON, I. PERUGIA AND D. SCHOTZAU, *hp-DGFEM for Maxwell's equations*, in Numerical Mathematics and Advanced Applications: ENUMATH 2001, F. Brezzi, A. Bua, S. Corsaro and A. Murli, eds., Springer-Verlag, Berlin, pp. 785-794, 2003.
- [8] P. HOUSTON, I. PERUGIA AND D. SCHOTZAU, *Mixed discontinuous Galerkin approximation of the Maxwell operator*, SIAM J. Numer. Anal., Vol. 42, pp. 434-459, 2004.
- [9] P. MONK, *Finite element methods for Maxwell's equations*, Oxford University Press, New York, 2003.
- [10] L. MU, J. WANG AND X. YE, *Weak Galerkin finite element methods on polytopal meshes*, arXiv:1204.3655v2, International Journal of Numerical Analysis and Modeling, Vol. 12, Number 1, pp. 31-53, 2015.
- [11] L. MU, J. WANG, X. YE AND S. ZHANG, *Weak Galerkin finite element method for the Maxwell equations*, arXiv:1312.2309, Journal of Scientific Computing, Vol. 65, Issue 1, pp. 363-386, 2015.
- [12] J. NÉDÉLEC, *Mixed finite elements in  $R^3$* , Numer. Math., Vol. 35, pp. 315-341, 1980.
- [13] I. PERUGIA AND D. SCHOTZAU, *The hp-local discontinuous Galerkin method for low-frequency time-harmonic Maxwell equations*, Math. Comp., Vol. 72, pp. 1179-1214, 2003.
- [14] I. PERUGIA, D. SCHOTZAU AND P. MONK, *Stabilized interior penalty methods for the timeharmonic Maxwell equations*, Comput. Methods Appl. Mech. Engrg., Vol. 191, pp. 4675-4697, 2002.
- [15] C. WANG AND J. WANG, *Discretization of div-curl systems by weak Galerkin finite element methods on polyhedral partitions*, arXiv: 1501.04616, Journal of Scientific Computing, Vol. 68, Issue 3, pp. 1144-1171, 2016.
- [16] C. WANG, J. WANG, R. WANG AND R. ZHANG, *A Locking-Free Weak Galerkin Finite Element Method for Elasticity Problems in the Primal Formulation*, arXiv:1508.03855, Journal of Computational and Applied Mathematics, Available online 31 December 2015. <http://www.sciencedirect.com/science/article/pii/S0377042715006275>.
- [17] C. WANG AND J. WANG, *A hybridized formulation for weak Galerkin finite element methods for biharmonic equation on polygonal or polyhedral meshes*, International Journal of Numerical Analysis and Modeling, Vol.12, pp. 302-317, 2015.
- [18] C. WANG AND J. WANG, *An efficient numerical scheme for the biharmonic equation by weak Galerkin finite element methods on polygonal or polyhedral meshes*, Journal of Computers and Mathematics with Applications, Vol. 68, Number 12, pp. 2314-2330, 2014.
- [19] J. WANG AND X. YE, *A weak Galerkin mixed finite element method for second-order elliptic problems*, arXiv:1202.3655v1, Math. Comp., Vol. 83, Number 289, pp. 2101-2126, 2014.
- [20] J. WANG AND X. YE, *A weak Galerkin finite element method for the Stokes equations*, arXiv:1302.2707, Advances in Computational Mathematics, Vol. 42, Issue 1, pp. 155-174, 2016.
- [21] J. WANG AND X. YE, *A weak Galerkin finite element method for second-order elliptic problems*, arXiv:1104.2897, J. Comp. and Appl. Math, Vol. 241, pp. 103-115, 2013.